Integrating $x\ dx$

By Ed Meyer

Students in a class on integral calculus will "know" that the integral of $x\ dx$ is $x^2/2$. However, knowing that the integral of $x\ dx$ is $x^2/2$ is not that useful outside calculus class unless you understand what it means. After all, you can teach a three-year old to respond "$x^2/2$" when asked the integral of $x\ dx$.

Let's think about this integral in terms of a real-world problem so we can understand it.

When a spring is stretched, the force it exerts is kx where k is the stiffness of the spring (assumed constant over the stretch) and x is the distance stretched.

The energy stored in the spring is the force the spring exerts multiplied by the stretch. However, the force the spring exerts is not constant. It is relatively easy to stretch the spring in the beginning, and it becomes harder to stretch it the more it is stretched.

The way to get the exact amount of energy stored in a stretched spring is to break up the total stretch into a large number of small stretches and calculate the increase in the energy stored in the spring for each stretch and then sum up all the energies.

The reason why the stretch has to be small is that the force the spring exerts can be considered constant over a infinitesimally small stretch. In the language of calculus, this is written

$$\int (kx)\ dx$$

The force the spring exerts is kx, and the distance it is stretched is the infinitesimal distance dx. The curved line in the equation above indicates that the sum is taken over all the x, and there are an infinite number of them.

Let's do it. We should start by really understanding what is going on. We can write the sum like this.

$$(kx_1)dx + (kx_2)dx + (kx_3)dx + (kx_4)dx + \cdots$$

Each term represents the increase in the energy stored in the spring. Each stretch is the same size, dx, and the force the spring exerts is the spring constant k multiplied by the distance the spring is stretched.

We can factor out a k from this sum to get

$$k[(x_1)dx + (x_2)dx + (x_3)dx + (x_4)dx + \cdots]$$

We can also factor out the dx. This leaves

$$k\ dx[(x_1) + (x_2) + (x_3) + (x_4) + \cdots]$$

Since each pull is dx long, the distance the spring is stretched increases by dx each time. So, $x_1 = dx$, $x_2 = 2dx$, $x_3 = 3dx$…. We now can write

$$k\ dx[dx + 2\ dx + 3\ dx + 4\ dx + \cdots]$$

We can factor out a dx again to get

$$k\ dx^2[1 + 2 + 3 + 4 + \cdots]$$

Now we have the problem of summing up the integers from 1 to a very large number, N. The average value of the integers from 1 to N is $N/2$ and there are N of them, so their sum is $N^2/2$.

Substituting this expression back into the equation for the energy stored in a stretched spring, we get

$$k\ dx^2[N^2/2]$$

We don't know the value of dx or N. The value of dx is infinitesimally small and N is infinitely large. However, we know that their product must be the total stretch, which we'll call X. After all, we divided the stretch into N pieces that are dx long. As a result, $(N^2dx^2) = X^2$.

This means that the energy stored in a spring that is stretched a distance X is

$$\frac{1}{2}kX^2$$

and that the integral of $x\ dx$ is $x^2/2$.

Quote Acrostic

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Instructions: Fill in the words at the bottom from the clues. Then write those letters in the grid at the top to reveal a quote. Black squares indicate the end of a word and punctuation has been removed. When you're done the first letters of the answers to the clues will be the subject of the quote.

QUOTE

1A	2B	3C	4D	5I	6E	7F	8H	9C		10B	11E		12H	13F	14C		15E	16B	17A	
18G	19D	20A	21F	22H	23I	24G		25B	26A		27E		28F	29E	30C	31D		32E	33I	
34F	35G		36B	37H	38I		39C	40A	41B	42D	43F	44D	45E	46B		47H	48A		49H	
50B	51G	52F	53I																	

CLUES

A. Tower in France

17	40	48	26	1	20		

B. Close combat between military aircraft

2	25	46	50	10	41	16	36

C. Birthday candles after being blown out

3	9	39	30	14			

D. Edible kind of dog

4	42	19	31	44			

E. Achieves a goal

29	15	6	27	32	45	11	

F. Marines' Hymn: "To the shores of _____"

43	52	7	28	13	21	34	

. Partner of "ands" and "buts"

51	18	35					

H. Cleveland resident

47	37	22	8	49	12		

I. Nullify

23	38	24	5	33	53		